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Spline Functions with Free Knots as the Limit of Varisolvent Families

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1. INTRODUCTION

In this paper we show that spline functions with free knots can be obtained as the limit of varisolvent families.[1] For a varisolvent family the Remez algorithm can be employed to obtain best uniform approximations. Thus an estimate to the best approximation using spline functions with free knots can be obtained from a sequence of best approximations gotten from an appropriate sequence of varisolvent families. One class of problems where this technique could be useful arises in optimal integration theory, [13, p. 45]. In this paper we deal with the uniform norm, but it should be noted that this technique would also be useful for other L_p norms [14].

The main difficulties in uniform spline approximation with free knots are caused by the fact that spline functions do not form Haar systems, and secondly, by the fact that when some of the knots coalesce, degrees of freedom are lost [1]. The first problem can be circumvented by standard perturbation techniques [5]. The second problem is more delicate and requires a new real parameterization which allows the perturbed knots to become possibly complex. The technique we use to solve this second problem is called extended varisolvence [1].

2. CHARACTERIZATION OF BEST APPROXIMATION

We begin the paper by stating a theorem on the characterization of best approximation in a family which is the limit of Haar systems.

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[1] Except for the terms Haar and varisolvent families [1], we use the definitions of Karlin [5].

For any function $g(x) \in C[0, 1]$, let $\| g \| = \max_{x \in [0,1]} | g(x)|$. By a best approximation we mean best approximation with respect to this norm. In this section we assume:

Basic Assumptions. Assume we are given N functions $s_i(x, \sigma)$ $i = 1,..., N$, $x \in [0, 1]$, and $\sigma \in [0, \sigma_0]$, which are continuous on $[0, 1] \times [0, \sigma_0]$ with the following two properties:

1. For fixed $\sigma > 0$, the N functions $s_i(x, \sigma)$ form a Haar system of order N.
2. The N functions $s_i(x, 0)$ are linearly independent.

Given any continuous function $f(x) \in C[0, 1]$, for each fixed $\sigma > 0$, let $S_f^*(\sigma) = \sum_{i=1}^{N} A_i(\sigma, f)\, s_i(x, \sigma)$ be the unique best approximation to $f(x)$ for $x \in [0, 1]$. See Rice [7, Chapts. 1, 3].

Further let $S(\sigma)$, $S_1(\sigma)$ stands for functions of the form $\sum_{i=1}^{N} A_i s_i(x, \sigma)$.

The classic Chebyshev theorem states that for a Haar system of order N, S^* is the best approximation to f if $f - S^*$ alternates N times; that is, there is a set of $N + 1$ points $0 \leqslant x_1 < \cdots < x_{N+1} \leqslant 1$ such that $\| f - S^* \| = | f(x_1) - S^*(x_1)|$ and $f(x_{i+1}) - S^*(x_{i+1}) = -(f(x_i) - S^*(x_i))$ $i = 1,..., N$. For such a limit family the N alternations are sufficient. For completeness we give a short proof of a theorem which can be found in [12].

THEOREM 1. *For each $f \in C[0, 1]$, there is a best approximation $S(0)$ so that $f - S(0)$ alternates N times.*

Proof. Let $\{\sigma_i\}$ be any sequence of positive numbers converging to zero. Since $\| f - S_f^*(\sigma_i)\| \leqslant \| f \|$ we have

$$\| S_f^*(\sigma_i)\| \leqslant 2 \| f \| . \tag{1}$$

By (1) and property 2 of the limit family it follows by a standard compactness argument that for some subsequence which we again call σ_i, a $S(0)$ exists so that

$$\lim_{i \to \infty} \| S_f^*(\sigma_i) - S(0)\| = 0.$$

Since each $f - S_f^*(\sigma_i)$ alternates N times it must follow that $f - S(0)$ does also. Let λ be the distance of f from the limit family. Clearly for each $\epsilon > 0$ there is an i_ϵ such that $i \geqslant i_\epsilon$ implies $\lambda + \epsilon \geqslant \| f - S(\sigma_i)\|$ for some $S(\sigma_i)$. Thus $\lambda \geqslant \| f - S(0)\|$. Hence $S(0)$ is a best approximation to f with the desired property.

Every best approximation does not necessarily alternate N times; indeed, a special case of Theorem 2 of [11] is that for a non-Haar system of order N there is an $f \in C\,[0, 1]$ with a best approximation S^* with the property that $f - S^*$ alternates at most $N - 1$ times. Splines form a non-Haar system [8].

3. Spline Functions as Limits of Haar Systems

We now establish that spline functions with fixed but possibly multiple knots can be considered limits of Haar systems.

Let $q_i(x)$ be positive and of class $C^{n-i}[0, 1]$, $i = 1,\dots, n$. We extend the $q_i(x)$ to be positive and of class C^{n-i} in $[-1, 2]$, $i = 1,\dots, n$. Let

$$\Phi_n(x, t) = \begin{cases} 0 & x < t \\ q_1(x) \int_t^x q_2(\zeta_1) \int_t^{\zeta_1} q_3(\zeta_2) \cdots \int_t^{\zeta_{n-1}} q_n(\zeta_{n-1})\, d\zeta_{n-1} \cdots d\zeta_1 & x \geqslant t. \end{cases}$$

Remark 1. $\Phi_n(x, t)$ for fixed t, as a function of x is a solution of differential equation $L_n\Phi = 0$, it is of continuity class C^{n-2}, while its $n - 1$ derivative has a jump of $1/q_n(t)$ at $x = t$. These properties determine $\Phi_n(x, t)$ uniquely. Conversely (see [3, prob. 22 p. 101]) for fixed x, $\Phi_n(x, t)$ as a function of t is a solution of the adjoint differential equation $L_n{}^+\Phi = 0$; it is of continuity class C^{n-2}, while its $n - 1$ derivative has a jump of $-1/q_n(x)$ at $x = t$.

Remark 2. Since the differential equations L_n and $L_n{}^+$ have essentially all the same properties, it follows that properties of $\Phi_n(x, t)$ as a function of x, are also true as a function of t.

Let

$$G_\sigma(w, y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(w - y)^2\right) \tag{2}$$

$$F_\sigma(w, x) = \int_{-1}^{2} G_\sigma(w, y)\, \Phi_n(x, y)\, dy. \tag{3}$$

Note that for fixed knots we permit only real w; but, in the free knot case we allow complex w.

Then it reasily follows from properties of the kernel $G_\sigma(w, y)$ (see, e.g., [4, pp. 156–157; 9, p. 65]) that:

A. For fixed w, $F_\sigma(w, x)$ is continuous for $x \in [0, 1]$, $\sigma \in [0, \sigma_0]$ any $\sigma_0 > 0$.

B. $\lim_{\sigma\to 0} \partial_j F_\sigma(w, x) = \partial_j \Phi_n(x, w)$, $j = 0,\dots, n - 2$; $\left(\partial_j \equiv \dfrac{\partial^j}{\partial w^j}\right)$.

(We remark that we extended the $q_i(x)$ beyond [0, 1], so that (A) and (B) would apply when $w = 0$ or 1).

C. For $\sigma > 0$, $F_\sigma(w, x)$ is analytic in w, and real for real w.

D. The proof in Karlin [5, pp. 512–513] shows that the kernel $F_\sigma(w, x)$

is strictly totally positive in x and extended totally positive in w. (It should be noted that when Karlin defines $F_\epsilon(x, i)$ he integrates $\Phi_n(x, y)$ with respect to x, not y as we have done. However, from Remark 1, it follows that Karlin's procedure carries over to the case considered in the present paper)

Finally, if we set $s_i(x, \sigma) = F_\sigma(w_i, x)$, $i = 1,..., n + k$, with

$$-1 < w_1 < w_2 < \cdots < w_n < 0 < w_{n+1} < \cdots < w_{n+k} < 1.$$

It follows from properties A, B, C, and D, that the $s_i(x, \sigma)$ satisfy the basic assumptions of Section 2. Furthermore, since the most general solution of the differential equation $L_n\Phi = 0$, for $x \in [0, 1]$ can be written as

$$\Phi = \sum_{i=1}^{n} A_i\Phi(x, w_i)$$

(as follows from Karlin [5, Theorem 1. 1, p. 503]), it follows that the above family is the setting of fixed simple knots treated by Schumaker [8], and Rice [7, Chap. 10]. Furthermore, if, for example, in the above family we have $w_{n+\alpha-1} < w_{n+\alpha} = w_{n+\alpha+1} = w_{n+\alpha+2} = \cdots = w_{n+\alpha+m} < w_{n+\alpha+m+1}$ and we set $s_{n+\alpha+q}(x, \sigma) = \partial_q F_\sigma(w_{n+\alpha}, x)$ $q = 1,..., m$, we are able to treat fixed multiple knots, as done recently by Braess [2]. In summary we may apply the results of Section 2 to obtain a result first given by Schumaker [8] and Jones, Karlowitz [12] (where the limiting process is used) for simple fixed knots.

THEOREM 2. *Consider the set of spline functions with r fixed knots w_{n+i}; $i = 1,..., r$, with multiplicity m_i where*

$$\sum_{i=1}^{r} m_i = k, \qquad 0 < w_{n+1} < \cdots < w_{n+r} < 1, \qquad m_i \leqslant n - 1.$$

Then there is a member s of this set which is a best approximation to f and for which $f - s$ alternates $n + k$ times.

The set of spline functions with fixed knots is of the form

$$\left\{\sum_{i=1}^{n} a_i\Phi(x, w_i) + \sum_{i=1}^{r}\sum_{j=0}^{m_j-1} b_{ij}\partial_j\Phi(x, w_{n+i}) : a_i, b_{ij} \text{ free real parameters}\right\}.$$

4. SPLINE FUNCTIONS WITH FREE KNOTS

In Section 3, we showed how splines with fixed knots may be viewed as limits of strictly totally positive families. In this section we wish to show how splines with variable knots may be viewed as limits of varisolvent families.

Before stating our main result, we must introduce some notation:

Let $\gamma(w, y)$ be an analytic function of $w = \sigma + i\tau$, for each $y \in [c, d]$, and real valued for real w. Denote $\partial^j\gamma(w, y)/\partial w^j$ by $\partial_j\gamma$. In the following assume

$$\begin{aligned} &w_\nu = \sigma_\nu & & \nu < r \\ &w_\nu = p_\nu + i\tau_\nu \quad & \tau_\nu \neq 0 \quad & \nu \geqslant r \\ &\sum_{\nu=1}^{r-1} m_\nu + 2\sum_{\nu=r}^{k} m_\nu = n & & \end{aligned} \tag{4}$$

and consider the subspace:

$$T_n[\sigma_1(m_1),..., \sigma_{r-1}(m_{r-1}), w_r(m_r),..., w_k(m_k); \gamma] \tag{5}$$

$$= \Big\{ \sum_{\nu=1}^{r-1} \sum_{\alpha=0}^{m_\nu-1} s_{\nu j}\partial_j\gamma(\sigma_\nu, y) + \sum_{\nu=r}^{k} \sum_{\alpha=0}^{m_\nu-1} [g_{\nu j} \operatorname{Re} F_{\nu j} + h_{\nu j} \operatorname{Im} F_{\nu j}];$$

$$s_{\nu j}, g_{\nu j}, h_{\nu j} \text{ real; } \alpha_{\nu j} \text{ real but fixed; and } F_{\nu j} = \exp(i\alpha_{\nu j})\, \partial_j\gamma(w_\nu, y) \Big\}.$$

Clearly with

$$A_{\nu j} = \begin{cases} s_{\nu j} & \nu < r \\ g_{\nu j} - ih_{\nu j} & \nu \geqslant r \end{cases}$$

and $\alpha_{\nu j} = 0$ for $\nu < r$, (5) can be written as

$$\sum_{\nu=1}^{k} \sum_{j=0}^{m_\nu-1} \operatorname{Re}[(A_{\nu j})(F_{\nu j})]. \tag{6}$$

In [1] we have proven the following theorem:

THEOREM 3. *Let Q be the subset of the complex plane where $|\operatorname{Im} w| < \pi/(d-c)$. Then if $\gamma(w, y) = \exp(wy)$ for $y \in [c, d]$, it is possible to choose real $\alpha_{\nu j}$ such that*

$$T_n[\sigma_1(m_1),..., \sigma_{r-1}(m_{r-1}), w_r(m_r),..., w_k(m_k), \gamma] \tag{7}$$

is a Haar subspace of dimension n for any n whenever the σ_j and w_j belong to Q.

For the present paper, we need the following additional theorem:

THEOREM 4. *Let Q be the subset of the complex plane where*

$$|\operatorname{Im} w| < \pi q/(2(d-c)).$$

Then if $\tilde{\gamma}(w, y) = \exp(-(w - y)^2/q)$ *for* $y \in [c, d]$, *it is possible to choose real* $\alpha_{\nu j}$ *such that*

$$T_n[\sigma_1(m_1),..., \sigma_{r-1}(m_{r-1}), w_r(m_r),..., w_k(m_k), \tilde{\gamma}] \tag{8}$$

is a Haar subspace of dimension n for any n whenever the σ_j *and* w_j *belong to* Q.

Proof. We use the device in [6, p. 11] of reducing Theorem 4 to Theorem 3. Consider, for example, the terms in (8) corresponding to $w_\nu(m_\nu)$. Using the notation of (6) with $B_{\nu j} = A_{\nu j} \exp(i\alpha_{\nu j})$ we have

$$\begin{aligned}
&\sum_{j=0}^{m_\nu-1} \operatorname{Re}[B_{\nu j}\partial_j[\tilde{\gamma}(w_\nu, y)]] \\
&\qquad = \exp(-y^2/q) \sum_{j=0}^{m_\nu-1} \operatorname{Re}[B_{\nu j}\partial_j[\exp(-w_\nu^2/q)]\, \gamma(2w_\nu/q, y)] \\
&\qquad = \exp(-y^2/q) \sum_{k=0}^{m_\nu-1} \operatorname{Re}[\tilde{B}_{\nu k}\partial_k\gamma(2w_\nu/q, y)]
\end{aligned} \tag{9}$$

with

$$\tilde{B}_{\nu k} = \sum_{j=k}^{m_\nu-1} B_{\nu j} \binom{j}{k} \partial_{j-k} \exp(-w_\nu^2/q); \tag{10}$$

thus,

$$\begin{aligned}
\tilde{B}_{\nu,m_\nu-1} &= \exp(-w_\nu^2/q)\, B_{\nu,m_\nu-1} \\
\tilde{B}_{\nu,m_\nu-2} &= \exp(-w_\nu^2/q)[B_{\nu,m_\nu-2} - (m_\nu - 1)(2w_\nu/q)\, B_{\nu,m_\nu-1}] \\
&\ \ \vdots \qquad\qquad\qquad \vdots
\end{aligned} \tag{11}$$

Thus if a nonzero element of (8) has n zeros, it follows from (9) and Theorem 3 that $\tilde{B}_{\nu k} = 0$. Further, it then follows from (11) by induction that $B_{\nu k} = 0$. Thus Theorem 4 is established.

Finally we have

THEOREM 5. *The class of spline functions of degree n, with k free knots, may be uniformly approximated by a sequence of varisolvent families each of maximal degree* $n + 2k$.

Proof. We establish two lemmas, from which the theorem immediately follows:

LEMMA 1. *There exists a constant K not depending on σ, such that, if Q_σ is the subset of the complex plane where* $|\operatorname{Im} w| < K\sigma$, *and if* $\gamma(w, y) = F_\sigma(w, y)$ *of* (3), $y \in [0, 1]$, *it is possible to choose real* $\alpha_{\nu j}$ *such that*

$$T_n[\sigma_1(m_1),..., \sigma_{r-1}(m_{r-1}), w_r(m_r),..., w_k(m_k), \gamma]$$

is a Haar subspace of dimension n for all n whenever the σ_j and w_j belong to Q_σ.

Proof. The lemma follows from Theorem 4, by again using the proof in Karlin [5, pp. 512–513], just as was done in D in Section 3.

LEMMA 2. *Let*

$$\tilde{F}_\sigma(a_j, c_j, e_j, g_j, y) = \frac{1}{2\pi i} \int_\Gamma \left[\frac{C(w)}{\prod_{i=1}^n (w - w_i) A(w)} + \sum_{j=1}^{k-p} \frac{g_j}{w - e_j}\right] F_\sigma(w, y)\, dw, \tag{12}$$

where $-1 < w_1 < w_2 < \cdots < w_n < 0$

$$A(w) = w^p + a_1 w^{p-1} + \cdots + a_p$$

$$C(w) = c_1 w^{p+n-1} + \cdots + c_{p+n}.$$

The numbers a_j, c_j, g_j, e_j are all real. The roots of $\prod_{i=1}^n (w - w_i) A(w)$ *are in the interior of the simple closed rectifiable contour Γ (which is in Q_σ) and are different from the $k - p$ distinct numbers $e_1,..., e_{k-p}$ which also are in the interior of Γ. Further, each root z of $A(w)$ satisfy real* $z \in (0, 1)$ *and* $|\operatorname{Im} z| < K\sigma$ *with K the constant of Lemma 1. Then*

1. $\tilde{F}_\sigma(a_j, c_j, e_j, 0, y)$ *is varisolvent of degree* $n + k + p$.
2. *If all the roots of $A(z)$ are real and are of multiplicity less than* $n - 1$,

then as $\sigma \downarrow 0$, $\tilde{F}_\sigma(a_j, c_j, e_j, 0, y)$ *uniformly approaches a spline function.*
Conversely, any spline function may be obtained in this manner.

Proof. Statement 1 follows from Theorem 1 of [1], applied to $\gamma = F_\sigma(w, y)$, with the properties established for it by Lemma 1 above.

Statement 2 follows from an application of the residue Theorem to (12) which yields,

$$\tilde{F}_\sigma(a_j, c_j, e_j, 0, y) = \sum_{i=1}^{n+p} B_i F_\sigma(w_i, y) \tag{13}$$

where if a root w_m occurs j times, replace $F_\sigma(w_{m+q}, y)$ by $\partial_q F_\sigma(w_m, y)$ $q = 1,..., j - 1$ in the sum in (13).

Then statement 2 follows from property B of Section 3.

The class of spline functions with free knots that results is of the form

$$\left\{ \sum_{i=1}^{n} a_i \Phi(x, w_i) + \sum_{i=1}^{r} \sum_{j=0}^{m_i-1} b_{ij} \partial_j \Phi(x, w_{n+i}); \right.$$

$$\sum_{i=1}^{r} m_i = k, \ -1 < w_1 < \cdots < w_n < 0 \text{ are fixed};$$

$$\left. m_i, a_i, b_{ij}, w_{n+i} \text{ are free real parameters} \right\}.$$

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